**Voronoi Diagrams: Fortune’s Algorithm**

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1. Introduction

A **Voronoi Diagram** is a way of breaking up a space into different regions. Voronoi diagrams are often used in city planning because they can efficiently set up a city with public services. Some commonly known applications of Voronoi diagrams are school districts, police and fire stations, and pizza delivery routes. In these examples, the schools, stations, and restaurants are considered nodes in which the region dedicated to that particular node is closer in distance to any other node in the city. In these examples, the city would be considered our space that is getting broken up into different regions. Voronoi Diagrams can also be found throughout nature, and, in particular, the patterns on certain animals show their presence in the world.

**Figure 1.** Berks County, Pennsylvania school districts map shows schools and regions surrounding schools forming a district.

![Voronoi Diagrams in nature](image1)

**Figure 2.** Examples of Voronoi diagrams in nature.

2. Voronoi Diagrams

**Definition 1.** Let \( P \) be a set of points in \( \mathbb{R}^2 \). The Voronoi diagram of \( P \), denoted \( V_P \), is a collection of Voronoi regions \( V_p \) for each point \( p \) in \( P \), where \( V_p = \{ x \in \mathbb{R}^2 : |x - p| \leq |x - q| \text{ for any } q \in P \} \). A point \( p \) is a as is on a Voronoi edge if it is equidistant between 2 nodes. A point \( p \) is a vertex if it is equidistant between 3 or more nodes.

We will now look at some examples of Voronoi diagrams in a plane.

**Figure 3.** This image shows the Voronoi diagrams between 2 points, 3 collinear points, and 3 non-collinear points. We can see that as the points become non-collinear and increase in number, the Voronoi diagram becomes more complicated.

**Figure 4.** This is the Voronoi diagram created by 19 nodes. The different regions are represented by the different colors shown.

Since the difficulty is steadily increasing, the ability to create these diagrams by hand becomes tedious and exhausting, so we can use an algorithm to make this task easier and more manageable.

3. Fortune’s Algorithm

**Fortune's Algorithm** Consider a point \( p \) and a line \( l \) that does not contain \( p \), then the set of points whose distance to \( p \) equals its distance to \( l \) forms a parabola. The point \( p \) is known as the focal point and \( l \) is known as the directrix of the parabola created. We will denote this parabola as \( P_{p,l} \).

**Definition 2.** Consider a plane with a set of points, \( P \). The sweep line is the line that moves through the plane from top to bottom passing through the points in \( P \). As the sweep line goes through the plane, we can create parabolas with the points in \( P \) above the sweep line being focal points and the sweep line being the directrix for these parabolas.

Now, consider a point \( q \) in \( P \) with coordinates \( q = (q_x, q_y) \) and denote the Euclidean distance between \( q \) and \( p \) as \( d(q, p) \). Also, since the sweep line is horizontal, we can define its vertical coordinate as \( y_q \), so \( d(q, l) = q_y - y_q \).

Thus, to say that \( q \) is on the parabola \( P_{p,l} \), we need the condition that \( d(q, l) = d(q, p) = q_y - y_q \). And we can more generally say that

\[
\begin{align*}
    d(q, p) &< q_y - y_q \quad \text{if } q \text{ lies above } P_{p,l}, \\
    d(q, p) &> q_y - y_q \quad \text{if } q \text{ lies below } P_{p,l}.
\end{align*}
\]

**Definition 3.** The Beach Line is the curve formed by the lowest parabolic arcs in the plane. As the sweep line moves through the plane, the beach line changes. Parabolas are being formed, and a vertical line will pass through several of them at once. The point at which the vertical line passes through the beach line is the lowest such point.

**Figure 5.** In the figure above, the beach line is green and the sweep line is purple.

4. Conclusion

Although there are multiple algorithms that can be used to create Voronoi diagrams, Fortune’s Algorithm is the most commonly used because it is quick and efficient to create Voronoi Diagrams. This algorithm becomes handy especially when dealing with many sites due to the tedious procedure needed to make these space-partitioning diagrams. Future research would include using different metrics to define distances, and then seeing how the Voronoi diagrams would be impacted by the change.

**References**
