Math Contest Level 2 - March 11, 2011

Coastal Carolina University

1. Beginning with 1, all the positive integers are written successively, beginning

$$1234567891011121314\ldots$$

What digit appears in the 2011^{th} position?

(a) 6 (b) 7 (c) 8 (d) 9 (e) 0

2. Given that $-2i$ is a root of the polynomial $p(x) = x^4 + 7x^3 + 16x^2 + 28x + 48$, the other three roots of p are:

(a) $-i, -3, -4$ (b) $-i, 3, 4$ (c) $2i, 3, 4$ (d) $2i, -3, -4$ (e) $2i, -2, -6$

3. The ratio of the area of a circle to the area of an inscribed square is

(a) $\frac{2}{\pi}$ (b) 2π (c) $\frac{\pi}{2}$ (d) $\frac{1}{\pi}$ (e) π

4. Evaluate $\tan \frac{\pi}{8}$.

(a) $\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$ (b) $\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$ (c) $\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}}$ (d) $\frac{\sqrt{2}}{2}$ (e) $\frac{1}{2}$

5. If the number 2011! were written in base 14, how many zeros would it end with?

(a) 287 (b) 328 (c) 333 (d) 334 (e) None of these

6. Let C be a cube where the length of its diagonal is the same as its volume. What is the length of each side?

(a) 1 (b) $2^{1/4}$ (c) $2^{1/3}$ (d) $3^{1/4}$ (e) None of these

7. Suppose x and y are real numbers such that $(x+5)^2 + (y-12)^2 = 196$. What is the minimum value of $x^2 + y^2$?

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 1 (d) 1/2 (e) None of these

8. Which of the following is/are true?

I. $\arcsin(-x) = -\arcsin x$

II. $\arccos(-x) = \pi - \arccos x$

III. $\arctan(-x) = \arctan x$

IV. $\sin(\arcsin x) = x$ provided $x \in [-1, 1]$

(a) I only
(b) I and III only
(c) II and IV only
(d) I, II, and IV only
(e) I, II, III, and IV

9. What is the smallest positive integer $n > 2010$ such that $\binom{n}{2011}$ is divisible by $\binom{n}{2010}$ but not equal to it?

(a) 2011 (b) 4021 (c) 6033 (d) 8043 (e) None of these

10. A circle is inscribed in a square. In any one corner of the square is an isosceles right triangle which shares a vertex with the square and has hypotenuse tangent to the circle. What is the ratio of the area of this triangle to the area of the square?

(a) $\dfrac{3-2\sqrt{2}}{4}$ (b) $\dfrac{3-2\sqrt{2}}{2}$ (c) $\dfrac{4-2\sqrt{2}}{3}$ (d) $\dfrac{4-3\sqrt{2}}{4}$ (e) None of these

11. The sum of a certain number of positive integers is 31. The largest value that their product can be is

(a) 78,672 (b) 80,448 (c) 78,748 (d) 80,484 (e) 78,732

12. Given that $1+\sqrt{2}$ is a root of $p(x) = x^3 + bx^2 + cx + 1$, where b and c are rational, what is the value of $b+c$?

(a) -3 (b) -2 (c) -1 (d) 0 (e) None of these

13. Let x be a real number. If $\csc x + \cot x = 2$, evaluate $\csc x - \cot x$.

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{2}}{4}$ (c) $\frac{1}{4}$ (d) $2\sqrt{3}$ (e) None of these

14. Let M be a real number such that the inequality

$$\sqrt{x-3} + \sqrt{6-x} \geq M$$

has a solution. The maximum value of M is

(a) $\sqrt{6} - \sqrt{3}$ (b) $\sqrt{3}$ (c) $\sqrt{6} + \sqrt{3}$ (d) $\sqrt{6}$ (e) None of these

15. Find the value of $\frac{1}{\log_2(2011!)} + \frac{1}{\log_3(2011!)} + \cdots + \frac{1}{\log_{2011}(2011!)}$.

(a) 1 (b) 2011 (c) $\ln(2011)$ (d) $\ln(2)$ (e) None of these

16. The solution set of the inequality $|x^2 - 2x - 2| > |x^2 - 2x + 2|$ is

(a) $(-2, 0)$ (b) $(-4, -2)$ (c) $(0, 3/2)$ (d) $(0, 4)$ (e) None of these

17. Suppose that A, B and C are the vertices of a triangle such that $|\overline{AB}| = 6, |\overline{BC}| = 8$, and $|\overline{AC}| = 10$. Two circles of equal radii are tangent to each other and two sides of the triangle. What is the common diameter of the two circles?

(a) $2\sqrt{3}$ (b) $\frac{20}{7}$ (c) $\frac{12}{5}$ (d) 3 (e) None of these

18. Consider an acute angle α such that the equation

$$\cot\alpha + 4x\cos\alpha + x^2 = 0$$

has a repeated root with respect to the variable x. Then the angle α, in radians, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ (c) $\frac{\pi}{6}$ or $\frac{5\pi}{12}$ (d) $\frac{\pi}{12}$ (e) None of these

19. Let A and B denote the points of intersection of the circles $x^2 + y^2 - 6x + 4y = 3$ and $x^2 + y^2 + 4x - 4y = 17$. What is the slope of segment AB?

(a) $\frac{5}{4}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$ (e) None of these

20. At the grocery store, Kristen bought 2 apples, 3 bananas and 4 cantaloupes for \$8. Justin bought 1 apple, 2 bananas and 1 cantaloupe for \$3 and Charlie bought 5 apples, 1 banana and 3 cantaloupes for \$9. How much would 1 apple, 1 banana, and 1 cantaloupe cost?

(a) \$2.00 (b) \$2.25 (c) \$2.50 (d) \$2.75 (e) None of these

21. If $f(x) + 2f(1-x) = x^2$ for all x, then $3f(x) =$

(a) x^2 (b) $x^2 - 4x + 2$ (c) $x^2 - 2x + 1$ (d) $x^2 + 2x - 1$ (e) None of these

22. An investor with \$20,000 wants to invest in four different mutual funds. The minimum investment in each mutual fund is 2, 2, 3 and 4 thousand dollars. How many investment strategies are possible if an investment must be made in each mutual fund (assume investments are made in units of one thousand dollars)?

(a) 220 (b) 230 (c) 240 (d) 250 (e) None of these

23. Solve the inequality

$$\sqrt{\log_2 x - 1} + \frac{1}{2}\log_{\frac{1}{2}} x^3 + 2 > 0.$$

(a) $[2,3)$ (b) $(2,3]$ (c) $[2,4)$ (d) $(2,4]$ (e) None of these

24. Three spheres of radius 1 are pairwise tangent and resting on a horizontal table. A fourth sphere of the same size is placed on top so that it lies tangent with the other three. How high is the top of the fourth sphere from the surface of the table?

(a) $2\sqrt{3}$ (b) $2+2\sqrt{3}$ (c) $\frac{2\sqrt{3}+2\sqrt{2}}{\sqrt{3}}$ (d) $\frac{4\sqrt{2}}{\sqrt{3}}$ (e) None of these

25. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|.$$

(a) $2\sqrt{2} - 1$ (b) $2\sqrt{2} + 1$ (c) $\sqrt{2} - 2$ (d) $\sqrt{2} + 2$ (e) None of these