Polar Notes

Coordinate Conversion

<table>
<thead>
<tr>
<th>Polar to Rectangular</th>
<th>Rectangular to Polar</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = r \cos \theta$</td>
<td>$r^2 = x^2 + y^2$</td>
</tr>
<tr>
<td>$y = r \sin \theta$</td>
<td>$\tan \theta = y/x$</td>
</tr>
</tbody>
</table>

Slope in Polar Form

If $f$ is a differentiable function of $\theta$, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point $(r, \theta)$ is

$$
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}
$$

provided that $dx/d\theta \neq 0$ at $(r, \theta)$.

Tangent Lines at the Pole

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Area in Polar Coordinates

If $f$ is continuous and nonnegative on the interval $[\alpha, \beta]$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$
A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta
$$

Arc Length of a Polar Curve

Let $f$ be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$
s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta
$$

Area of a Surface of Revolution

Let $f$ be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows

1. $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta$ \quad About the polar axis

2. $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta$ \quad About the line $\theta = \frac{\pi}{2}$
Useful Integrals

\( a \) is a constant

\[
\int \cos^2(a\theta) \, d\theta = \frac{1}{2} \theta + \frac{1}{4a} \sin(2a\theta) + C
\]

\[
\int \sin^2(a\theta) \, d\theta = \frac{1}{2} \theta - \frac{1}{4a} \sin(2a\theta) + C
\]

\[
\int \sqrt{1 + \cos(a\theta)} \, d\theta = \frac{2\sqrt{2}}{a} \sin \left( \frac{a\theta}{2} \right) + C, \quad \cos \left( \frac{a\theta}{2} \right) \geq 0
\]

\[
\int \sqrt{1 - \cos(a\theta)} \, d\theta = -\frac{2\sqrt{2}}{a} \cos \left( \frac{a\theta}{2} \right) + C, \quad \sin \left( \frac{a\theta}{2} \right) \geq 0
\]

\[
\int \sqrt{1 + \sin(a\theta)} \, d\theta = -\frac{2}{a} \sqrt{1 + \sin(a\theta)} + C, \quad \cos(a\theta) \geq 0
\]

\[
\int \sqrt{1 - \sin(a\theta)} \, d\theta = \frac{2}{a} \sqrt{1 + \sin(a\theta)} + C, \quad \cos(a\theta) \geq 0
\]

Special Polar Graphs

**Limaçons**

\[ r = a \pm b \cos \theta \]
\[ r = a \pm b \sin \theta \]

\( a > 0, b > 0 \)

\( \frac{a}{b} < 1 \) \hspace{1cm} \( \frac{a}{b} = 1 \) \hspace{1cm} \( 1 < \frac{a}{b} < 2 \) \hspace{1cm} \( \frac{a}{b} \geq 2 \)

- Limaçon with inner loop
- Cardioid (heart-shaped)
- Dimpled limaçon
- Convex limaçon

**Rose Curves**

\( a \) petals if \( n \) is odd

2\( n \) petals if \( n \) is even

\( n \geq 2 \)

- \( a \): Distance from polar axis to the tip of the petal

\[ r = a \cos n \theta \]
\[ (n = 3) \]

\[ r = a \sin n \theta \]
\[ (n = 4) \]

\[ r = a \cos n \theta \]
\[ (n = 5) \]

\[ r = a \sin n \theta \]
\[ (n = 2) \]

**Circles**

\( a \): Diameter of the circle

**Lemniscates**

\( a \): Distance from the pole to the tip of a loop

\[ r = a \cos \theta \]

Circle

\[ r = a \sin \theta \]

Circle

\[ r^2 = a^2 \sin 2\theta \]

Lemniscate

\[ r^2 = a^2 \cos 2\theta \]

Lemniscate